

# Non-minimal Einstein-Yang-Mills-Higgs theory: Associated, color and color-acoustic metrics for the Wu-Yang monopole model

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We discuss a non-minimal Einstein-Yang-Mills-Higgs model with uniaxial anisotropy in the group space associated with the Higgs field. We apply this theory to the problem of propagation of color and color-acoustic waves in the gravitational background related to the non-minimal regular Wu-Yang monopole.

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## I. INTRODUCTION

Non-minimal (induced by curvature) interactions of gravity field with electromagnetic, gauge and scalar fields (see, e.g., [1, 2] for review and basic references) are known to introduce specific coupling constants, which can be associated with radii of new (non-minimal) horizons, when one deals with stationary spherically symmetric objects [3, 4, 5]. The causal structure of spacetime surrounding the charged objects non-minimally coupled to gravity field is more sophisticated than the standard (minimal) one. To illustrate this circumstance one can mention recently obtained exact solutions of the three-parameter non-minimal Einstein-Yang-Mills model for the Wu-Yang monopole [6] and Wu-Yang wormhole [7]. At the same time these instances show explicitly that an appropriate choice of constants of non-minimal coupling,  $q_1$ ,  $q_2$  and  $q_3$ , can result not only in appearance of new (non-minimal) horizons, but in disappearance of singularities as well. For example, when  $q_1 = -q$ ,  $q_2 = 4q$ ,  $q_3 = -6q$  and  $q$  is positive, metric of the non-minimal Wu-Yang monopole (see [6], Eq.(40)) is regular at the center  $r = 0$  and has no horizons, when the mass of the object is less than the critical one,  $M_{\text{(crit)}}$ . In the presented paper we show that there exists a simple extension of this regular solution to the case, when the non-minimal Wu-Yang monopole possesses scalar Higgs field, which is covariantly constant and parallel to the Yang-Mills field strength in the group space. Considering the gravitational, gauge and scalar fields of such monopole as a non-minimal background, we start now to study the dynamics of *test* particles, quasi-particles and waves in the monopole environment. Despite the background gravity field is *regular*, the interaction of particles and waves with curvature (which can be reformulated in terms of effective non-minimal forces) results in the appearance of singularities of a dynamic type, analogous to the ones well-known in the Analogue Gravity theory [8, 9, 10]. For instance, particles, quasi-particles and waves can fall within trapped regions in the environment of the non-minimal Wu-Yang monopole. The main instrument of analysis of such a problem is the theory of effective metrics (see, e.g., the review [8]). It is well known, that optical metrics are alternative tools to investigate the propagation of light in the framework of electrodynamics of moving media (see [11, 12] and, e.g., [13, 14] for a review), the acoustic metrics give an alternative description of propagation of scalar waves in the framework of hydrodynamics [8, 9, 10]. When we deal with Yang-Mills-Higgs theory the *color* metrics appear instead of optical ones, and *color-acoustic* metrics can be introduced for the description of the Higgs scalar waves. The study and physical interpretation of singularities, appearing in color and color-acoustic metrics in the background of the regular non-minimal Wu-Yang monopole solution, is the main goal of the presented paper.

The paper is organized as follows. Section II contains description of the seven-parameter non-minimal Einstein-Yang-Mills-Higgs (EYMH) model: Subsection II A includes basic definitions, in Subsection II B we introduce uniaxial structure in the group space and in Subsection II C we obtain non-minimal master equations for the Yang-Mills, Higgs and gravity fields. Section III is devoted to the description of the exact solution of presented model, supplemented by the Wu-Yang ansatz [15]. In Section IV we study associated metrics (IV A), color metrics (IV B) and color-acoustic metrics (IV C) in the background of regular solution for the non-minimal Wu-Yang monopole. In Discussion we

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consider in more details the admissible and trapped regions for longitudinal and transversal color and color-acoustic waves.

## II. SEVEN-PARAMETER NON-MINIMAL EYMH MODEL

### A. Lagrangian

Consider a modified action functional<sup>1</sup>

$$S_{(\text{NMEYMH})} = \int d^4x \sqrt{-g} \left\{ \frac{R}{8\pi} + \frac{1}{2} F_{ik}^{(a)} F_{(a)}^{ik} - \hat{D}_m \Phi^{(a)} \hat{D}^m \Phi_{(a)} + \frac{1}{2} \lambda (\Phi^2 - v^2)^2 + \right. \\ \left. + \frac{1}{2} \mathcal{R}^{ikmn} X_{(a)(b)} F_{ik}^{(a)} F_{mn}^{(b)} - \mathfrak{R}^{mn} Y_{(a)(b)} (\hat{D}_m \Phi)^{(a)} (\hat{D}_n \Phi)^{(b)} \right\}, \quad (1)$$

where the susceptibility tensors  $\mathcal{R}^{ikmn}$  and  $\mathfrak{R}^{mn}$  are defined as

$$\mathcal{R}^{ikmn} \equiv \frac{q_1}{2} R (g^{im} g^{kn} - g^{in} g^{km}) + \frac{q_2}{2} (R^{im} g^{kn} - R^{in} g^{km} + R^{kn} g^{im} - R^{km} g^{in}) + q_3 R^{ikmn}, \quad (2)$$

$$\mathfrak{R}^{mn} \equiv q_4 R g^{mn} + q_5 R^{mn}. \quad (3)$$

Here  $g = \det(g_{ik})$  is the determinant of a metric tensor  $g_{ik}$ ,  $R$  is the Ricci scalar, tensor  $F_{ik}^{(a)}$  is the strength of gauge field, the symbol  $\Phi^{(a)}$  denotes the multiplet of the Higgs scalar fields,  $\frac{1}{2} \lambda (\Phi^2 - v^2)^2$  is a potential of the Higgs field, and  $\Phi^2 \equiv \Phi^{(a)} \Phi_{(a)}$ . Latin indices without parentheses run from 0 to 3,  $(a)$  and  $(b)$  are the group indices. Following [16], Section 4.3, we consider the Yang-Mills field  $\mathbf{F}_{mn}$  and the Higgs field  $\Phi$  taking values in the Lie algebra of the gauge group  $SU(n)$ :

$$\mathbf{F}_{mn} = -i\mathcal{G} \mathbf{t}_{(a)} F_{mn}^{(a)}, \quad \mathbf{A}_m = -i\mathcal{G} \mathbf{t}_{(a)} A_m^{(a)}, \quad \Phi = \mathbf{t}_{(a)} \Phi^{(a)}. \quad (4)$$

Here  $\mathbf{t}_{(a)}$  are Hermitian traceless generators of  $SU(n)$  group,  $F_{mn}^{(a)}$ ,  $A_i^{(a)}$  and  $\Phi^{(a)}$  are real fields ( $A_i^{(a)}$  represents the Yang-Mills field potential),  $\mathcal{G}$  is a constant of gauge interaction, and the group index  $(a)$  runs from 1 to  $n^2 - 1$ . The scalar product of the generators  $\mathbf{t}_{(a)}$  and  $\mathbf{t}_{(b)}$  is defined via the trace

$$(\mathbf{t}_{(a)}, \mathbf{t}_{(b)}) \equiv 2\text{Tr } \mathbf{t}_{(a)} \mathbf{t}_{(b)} \equiv G_{(a)(b)}, \quad (5)$$

the symmetric tensor  $G_{(a)(b)}$  plays a role of a metric in the group space. The representation (4), (5) allows to consider the multiplets of the real fields  $\{F_{mn}^{(a)}\}$ ,  $\{A_i^{(a)}\}$  and  $\{\Phi^{(a)}\}$  as components of the corresponding vectors in the  $n^2 - 1$  dimensional group space. The operations with the group indices  $(a)$  are the following: the repeating indices denote the convolution, and the rule  $\Phi_{(a)} = G_{(a)(b)} \Phi^{(b)}$  for the indices lowering takes place. The Yang-Mills fields  $F_{mn}^{(a)}$  are connected with the potentials of the gauge field  $A_i^{(a)}$  by the well-known formula [16]

$$F_{mn}^{(a)} = \nabla_m A_n^{(a)} - \nabla_n A_m^{(a)} + \mathcal{G} f_{(b)(c)}^{(a)} A_m^{(b)} A_n^{(c)}. \quad (6)$$

Here  $\nabla_m$  is a covariant space-time derivative, the symbols  $f_{(b)(c)}^{(a)}$  denote the real structure constants of the gauge group  $SU(n)$ . The gauge invariant derivative  $\hat{D}_m \Phi^{(a)}$  is defined according to the formula [16]

$$\hat{D}_m \Phi^{(a)} \equiv \nabla_m \Phi^{(a)} + \mathcal{G} f_{(b)(c)}^{(a)} A_m^{(b)} \Phi^{(c)}. \quad (7)$$

The definition of the commutator is based on the relation:

$$[\mathbf{t}_{(a)}, \mathbf{t}_{(b)}] = i f_{(a)(b)}^{(c)} \mathbf{t}_{(c)}, \quad (8)$$

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<sup>1</sup> Hereafter we use the units  $c = G = \hbar = 1$ .

providing the formula

$$f_{(c)(a)(b)} \equiv G_{(c)(d)} f_{(a)(b)}^{(d)} = -2i \operatorname{Tr} [\mathbf{t}_{(a)}, \mathbf{t}_{(b)}] \mathbf{t}_{(c)}. \quad (9)$$

The structure constants  $f_{(a)(b)(c)}$  are supposed to be antisymmetric under exchange of any two indices. The quantities  $X_{(a)(b)}$  and  $Y_{(a)(b)}$  are symmetric with respect to the group indices and depend on the Higgs field  $\Phi^{(c)}$ ; they are defined below.

### B. Higgs field and uniaxial structures in the group space

The group space can be considered as isotropic, when it is equipped by the metric  $G_{(a)(b)}$  only, and as anisotropic one in all other cases. The presence of the Higgs field multiplet  $\Phi^{(a)}$  with positive norm  $G_{(a)(b)} \Phi^{(a)} \Phi^{(b)} \equiv \Phi^2 > 0$  allows us to equip the group space by the vector  $q^{(a)} = \Phi^{(a)}/\Phi$  with the unit norm  $G_{(a)(b)} q^{(a)} q^{(b)} = 1$  and by the projector  $P_{(a)(b)}$ . The latter is defined as

$$P_{(a)(b)} \equiv G_{(a)(b)} - \frac{\Phi_{(a)} \Phi_{(b)}}{\Phi^2} = G_{(a)(b)} - q_{(a)} q_{(b)}, \quad (10)$$

and possesses the evident projector properties:

$$P_{(a)(b)} = P_{(b)(a)}, \quad P_{(a)(b)} \Phi^{(b)} = 0, \quad P_{(a)(b)} P^{(a)(c)} = P_{(b)}^{(c)}, \quad P_{(a)}^{(a)} = n^2 - 2. \quad (11)$$

Using these quantities, every tensor  $\mathcal{A}_{(a)(b)}^{i_1 i_2 \dots i_s}$ , symmetric with respect to the transposition of the group indices  $(a)$  and  $(b)$ , can be decomposed into the sum

$$\mathcal{A}_{(a)(b)}^{i_1 i_2 \dots i_s} = \mathcal{A}_{(\text{long})}^{i_1 i_2 \dots i_s} q_{(a)} q_{(b)} + \mathcal{B}_{(a)}^{i_1 i_2 \dots i_s} q_{(b)} + \mathcal{B}_{(b)}^{i_1 i_2 \dots i_s} q_{(a)} + \mathcal{D}_{(a)(b)}^{i_1 i_2 \dots i_s}, \quad (12)$$

where

$$\mathcal{A}_{(\text{long})}^{i_1 i_2 \dots i_s} \equiv \mathcal{A}_{(c)(d)}^{i_1 i_2 \dots i_s} q^{(c)} q^{(d)}, \quad \mathcal{D}_{(a)(b)}^{i_1 i_2 \dots i_s} \equiv \mathcal{A}_{(c)(d)}^{i_1 i_2 \dots i_s} P_{(a)}^{(c)} P_{(b)}^{(d)}, \quad (13)$$

$$\mathcal{B}_{(a)}^{i_1 i_2 \dots i_s} \equiv \mathcal{A}_{(c)(d)}^{i_1 i_2 \dots i_s} P_{(a)}^{(c)} q^{(d)} = \mathcal{A}_{(c)(d)}^{i_1 i_2 \dots i_s} q^{(c)} P_{(a)}^{(d)}, \quad (14)$$

$$\mathcal{B}_{(a)}^{i_1 i_2 \dots i_s} q^{(a)} = 0, \quad \mathcal{D}_{(a)(b)}^{i_1 i_2 \dots i_s} q^{(b)} = 0 = \mathcal{D}_{(a)(b)}^{i_1 i_2 \dots i_s} q^{(a)}. \quad (15)$$

The simplest type of anisotropy in the group space is the uniaxial one. This case assumes that the only direction pointed by  $q^{(a)}$  is the selected one, while other orthogonal directions are equivalent, i.e.,

$$\mathcal{A}_{(a)(b)}^{i_1 i_2 \dots i_s} q^{(b)} = \mathcal{S}_1^{i_1 i_2 \dots i_s} q_{(a)}, \quad \mathcal{A}_{(a)(b)}^{i_1 i_2 \dots i_s} P^{(b)(c)} = \mathcal{S}_2^{i_1 i_2 \dots i_s} P_{(a)}^{(c)}, \quad (16)$$

where  $\mathcal{S}_1^{i_1 i_2 \dots i_s}$  and  $\mathcal{S}_2^{i_1 i_2 \dots i_s}$  are some non-equal tensors without group indices. The conditions (16) mean that

$$\mathcal{B}_{(a)}^{i_1 i_2 \dots i_s} = 0, \quad \mathcal{D}_{(a)(b)}^{i_1 i_2 \dots i_s} = \mathcal{D}_{(\text{trans})}^{i_1 i_2 \dots i_s} P_{(a)(b)}, \quad \mathcal{D}_{(\text{trans})}^{i_1 i_2 \dots i_s} = \frac{1}{(n^2 - 2)} \mathcal{A}_{(a)(b)}^{i_1 i_2 \dots i_s} P^{(a)(b)}. \quad (17)$$

Keeping in mind this features, we assume the tensors  $X_{(a)(b)}$  and  $Y_{(a)(b)}$  in the Lagrangian (1) to have the simplest form

$$X_{(a)(b)} \equiv G_{(a)(b)} + (Q_1 - 1) \frac{\Phi_{(a)} \Phi_{(b)}}{\Phi^2}, \quad Y_{(a)(b)} \equiv G_{(a)(b)} + (Q_2 - 1) \frac{\Phi_{(a)} \Phi_{(b)}}{\Phi^2}, \quad (18)$$

with two new coupling constants  $Q_1$  and  $Q_2$ . In principle, we can treat them as quantities, which depend on the vacuum value  $v$  of the scalar field. When  $Q_1 = Q_2 = 1$ , the tensors  $X_{(a)(b)}$  and  $Y_{(a)(b)}$  coincide with the metric  $G_{(a)(b)}$  in the group space. The introduction of  $Q_1 \neq 1$  and/or  $Q_2 \neq 1$  modifies the master equations for the gauge, scalar and gravitational fields. When  $Q_1 = 0$ ,  $X_{(a)(b)} = P_{(a)(b)}$ , and when  $Q_2 = 0$ ,  $Y_{(a)(b)} = P_{(a)(b)}$ . In general case

$X_{(a)(b)}$  (and  $Y_{(a)(b)}$ , correspondingly) consists of two parts, parallel to the direction pointed by the scalar field, and orthogonal to it, the longitudinal and transversal parts being, respectively

$$X_{(\text{long})} = X_{(a)(b)} q^{(a)} q^{(b)} = Q_1, \quad X_{(\text{trans})} = \frac{1}{(n^2 - 2)} X_{(a)(b)} P^{(a)(b)} = 1. \quad (19)$$

In addition, every vector in the group space, say,  $A_k^{(a)}$ , can be decomposed using longitudinal and transversal components as follows

$$A_k^{(a)} = q^{(a)} A_{k(\text{long})} + A_{k(\text{trans})}^{(a)}, \quad (20)$$

where

$$A_{k(\text{long})} \equiv A_k^{(a)} q_{(a)}, \quad A_{k(\text{trans})}^{(a)} \equiv P_{(b)}^{(a)} A_k^{(b)}. \quad (21)$$

### C. Non-minimal master equations

#### 1. Non-minimal extension of the Yang-Mills field equations

The variation of the action functional over the Yang-Mills potential  $A_i^{(a)}$  yields

$$\hat{D}_k \mathcal{H}_{(a)}^{ik} = \mathcal{G}(\hat{D}_k \Phi^{(d)}) f_{(a)(h)}^{(b)} \Phi^{(h)} [G_{(b)(d)} g^{ik} + Y_{(b)(d)} \mathfrak{R}^{ik}], \quad (22)$$

where the tensor  $\mathcal{H}_{(a)}^{ik}$  is

$$\mathcal{H}_{(a)}^{ik} = C_{(a)(b)}^{ikmn} F_{mn}^{(b)} = \left[ \frac{1}{2} (g^{im} g^{kn} - g^{in} g^{km}) G_{(a)(b)} + \mathcal{R}^{ikmn} X_{(a)(b)} \right] F_{mn}^{(b)}. \quad (23)$$

Thus, the linear response tensor  $C_{(a)(b)}^{ikmn}$  is now of the form

$$C_{(a)(b)}^{ikmn} \equiv \left[ \frac{1}{2} (g^{im} g^{kn} - g^{in} g^{km}) + \mathcal{R}^{ikmn} \right] G_{(a)(b)} + (Q_1 - 1) \frac{\Phi_{(a)} \Phi_{(b)}}{\Phi^2} \mathcal{R}^{ikmn}, \quad (24)$$

the longitudinal and transversal parts being, respectively

$$C_{(\text{long})}^{ikmn} = \left[ \frac{1}{2} (g^{im} g^{kn} - g^{in} g^{km}) + Q_1 \mathcal{R}^{ikmn} \right], \quad (25)$$

$$C_{(\text{trans})}^{ikmn} = \left[ \frac{1}{2} (g^{im} g^{kn} - g^{in} g^{km}) + \mathcal{R}^{ikmn} \right]. \quad (26)$$

Mention that  $C_{(\text{trans})}^{ikmn}$  can be obtained from  $C_{(\text{long})}^{ikmn}$  by the formal replacement  $Q_1 \rightarrow 1$ , and let us use this features below for the simplifications of the formulas. The color permittivity, impermeability and cross-effect tensors are also anisotropic in the group space

$$\varepsilon_{(a)(b)}^{im} \equiv 2 C_{(a)(b)}^{ikmn} U_k U_n = [\Delta^{im} + 2 \mathcal{R}^{ikmn} U_k U_n] G_{(a)(b)} + 2(Q_1 - 1) \mathcal{R}^{ikmn} U_k U_n \frac{\Phi_{(a)} \Phi_{(b)}}{\Phi^2}, \quad (27)$$

$$(\mu^{-1})_{(a)(b)}^{im} \equiv -2 {}^* C_{(a)(b)}^{ikmn} U_k U_n = [\Delta^{im} - 2 {}^* \mathcal{R}^{ikmn} U_k U_n] G_{(a)(b)} - 2(Q_1 - 1) {}^* \mathcal{R}^{ikmn} U_k U_n \frac{\Phi_{(a)} \Phi_{(b)}}{\Phi^2}, \quad (28)$$

$$\nu_{(a)(b)}^{im} \equiv 2 {}^* C^{ikmn} U_k U_n = {}^* \mathcal{R}^{ikmn} U_k U_n \left[ G_{(a)(b)} + (Q_1 - 1) \frac{\Phi_{(a)} \Phi_{(b)}}{\Phi^2} \right]. \quad (29)$$

Here the asterisk denotes the dualization procedure:

$${}^* W^{ikmn} = \frac{1}{2} \epsilon^{ikpq} W_{pq}{}^{mn}, \quad {}^* W^{*ikmn} = \frac{1}{2} \epsilon^{pqmn} {}^* W^{ik}{}_{pq}, \quad (30)$$

$U^i$  and  $\Delta^{im} \equiv g^{im} - U^i U^m$  are a unit timelike vector ( $U_i U^i = 1$ ) and the projection tensor, respectively. The corresponding longitudinal and transversal components of  $\varepsilon_{(a)(b)}^{im}$ ,  $(\mu^{-1})_{(a)(b)}^{im}$  and  $\nu_{(a)(b)}^{im}$  can be easily written in analogy with (25) and (26).

## 2. Non-minimal extension of the Higgs field equations

The variation of the action  $S_{(\text{NMEYMH})}$  over the Higgs scalar field  $\Phi^{(a)}$  yields

$$\begin{aligned} \hat{D}_m \left\{ [g^{mn} G_{(a)(b)} + \Re^{mn} Y_{(a)(b)}] \hat{D}_n \Phi^{(b)} \right\} = & -\lambda \Phi_{(a)} (\Phi^2 - v^2) - \\ & - \frac{(Q_1 - 1)}{2\Phi^2} \mathcal{R}^{ikmn} F_{ik}^{(c)} F_{mn}^{(b)} \Phi_{(b)} P_{(a)(c)} + \frac{(Q_2 - 1)}{\Phi^2} \Re^{mn} P_{(a)(c)} \Phi_{(b)} (\hat{D}_m \Phi^{(c)}) (\hat{D}_n \Phi^{(b)}). \end{aligned} \quad (31)$$

The tensor  $\mathcal{C}_{(a)(b)}^{ik}$ , introduced as

$$\mathcal{C}_{(a)(b)}^{ik} = [g^{ik} + \Re^{ik}] G_{(a)(b)} + (Q_2 - 1) \Re^{ik} \frac{\Phi_{(a)} \Phi_{(b)}}{\Phi^2}, \quad (32)$$

can also be decomposed into longitudinal and transversal components

$$\mathcal{C}_{(a)(b)}^{ik} = \mathcal{C}_{(\text{long})}^{ik} q_{(a)} q_{(b)} + P_{(a)(b)} \mathcal{C}_{(\text{trans})}^{ik}, \quad (33)$$

where

$$\mathcal{C}_{(\text{long})}^{ik} \equiv \mathcal{C}_{(a)(b)}^{ik} q^{(a)} q^{(b)}, \quad \mathcal{C}_{(\text{trans})}^{ik} \equiv \frac{1}{n^2 - 2} \mathcal{C}_{(a)(b)}^{ik} P^{(a)(b)}. \quad (34)$$

The definitions

$$\tilde{g}_{(\text{long})}^{ik} \equiv \mathcal{C}_{(\text{long})}^{ik} = g^{ik} + Q_2 \Re^{ik}, \quad (35)$$

$$\tilde{g}_{(\text{trans})}^{ik} \equiv \mathcal{C}_{(\text{trans})}^{ik} = g^{ik} + \Re^{ik}, \quad (36)$$

introduce two color-acoustic metrics for the colored scalar particles. Again,  $\tilde{g}_{(\text{trans})}^{ik}$  can be obtained from  $\tilde{g}_{(\text{long})}^{ik}$  by the formal replacement  $Q_2 \rightarrow 1$ .

## 3. Master equations for the gravitational field

The equations for the gravity field related to the action functional  $S_{(\text{NMEYMH})}$  are of the form

$$R_{ik} - \frac{1}{2} R g_{ik} = 8\pi T_{ik}^{(\text{eff})}, \quad (37)$$

where

$$T_{ik}^{(\text{eff})} = T_{ik}^{(YM)} + T_{ik}^{(H)} + T_{ik}^{(NM)}. \quad (38)$$

The first term  $T_{ik}^{(YM)}$ :

$$T_{ik}^{(YM)} \equiv \frac{1}{4} g_{ik} F_{mn}^{(a)} F_{(a)}^{mn} - F_{in}^{(a)} F_{k(a)}^n, \quad (39)$$

is a stress-energy tensor of pure Yang-Mills field, the second term:

$$T_{ik}^{(H)} \equiv -\frac{1}{2} g_{ik} \hat{D}_m \Phi^{(a)} \hat{D}^m \Phi_{(a)} + \hat{D}_i \Phi^{(a)} \hat{D}_k \Phi_{(a)} + \frac{1}{4} g_{ik} \lambda (\Phi^2 - v^2)^2 \quad (40)$$

relates to the standard stress-energy of the Higgs field. The non-minimal contributions enter the last tensor  $T_{ik}^{(NM)}$ , which may be represented as a sum of 5 items:

$$T_{ik}^{(NM)} \equiv q_1 T_{ik}^{(I)} + q_2 T_{ik}^{(II)} + q_3 T_{ik}^{(III)} + q_4 T_{ik}^{(IV)} + q_5 T_{ik}^{(V)}. \quad (41)$$

The definitions of these five tensors relate to the corresponding coupling constant  $q_1, q_2, \dots, q_5$ . The tensors  $T_{ik}^{(I)}, T_{ik}^{(II)}, \dots, T_{ik}^{(V)}$  are

$$T_{ik}^{(I)} = RX_{(a)(b)} \left[ \frac{1}{4} g_{ik} F_{mn}^{(a)} F^{mn(b)} - F_{im}^{(a)} F_k^{m(b)} \right] - \frac{1}{2} R_{ik} X_{(a)(b)} F_{mn}^{(a)} F^{mn(b)} + \\ + \frac{1}{2} \left[ \hat{D}_i \hat{D}_k - g_{ik} \hat{D}^l \hat{D}_l \right] \left[ X_{(a)(b)} F_{mn}^{(a)} F^{mn(b)} \right], \quad (42)$$

$$T_{ik}^{(II)} = -\frac{1}{2} g_{ik} \left[ \hat{D}_m \hat{D}_l \left( X_{(a)(b)} F^{mn(a)} F_n^{l(b)} \right) - R_{lm} X_{(a)(b)} F^{mn(a)} F_n^{l(b)} \right] - \\ - F^{ln(a)} X_{(a)(b)} \left( R_{il} F_{kn}^{(b)} + R_{kl} F_{in}^{(b)} \right) - \frac{1}{2} \hat{D}^m \hat{D}_m \left( X_{(a)(b)} F_{in}^{(a)} F_k^{n(b)} \right) + \\ + \frac{1}{2} \hat{D}_l \left[ \hat{D}_i \left( X_{(a)(b)} F_{kn}^{(a)} F^{ln(b)} \right) + \hat{D}_k \left( X_{(a)(b)} F_{in}^{(a)} F^{ln(b)} \right) \right] - R^{mn} X_{(a)(b)} F_{im}^{(a)} F_{kn}^{(b)}, \quad (43)$$

$$T_{ik}^{(III)} = \frac{1}{4} g_{ik} R^{mnl s} X_{(a)(b)} F_{mn}^{(a)} F_{ls}^{(b)} - \frac{3}{4} X_{(a)(b)} F^{ls(a)} \left( F_i^{n(b)} R_{knls} + F_k^{n(b)} R_{inls} \right) - \\ - \frac{1}{2} \hat{D}_m \hat{D}_n \left[ X_{(a)(b)} \left( F_i^{n(a)} F_k^{m(b)} + F_k^{n(a)} F_i^{m(b)} \right) \right], \quad (44)$$

$$T_{ik}^{(IV)} = \left( R_{ik} - \frac{1}{2} g_{ik} R \right) Y_{(a)(b)} (\hat{D}_m \Phi^{(a)}) (\hat{D}^m \Phi^{(b)}) + R Y_{(a)(b)} (\hat{D}_i \Phi^{(a)}) (\hat{D}_k \Phi^{(b)}) + \\ + \left( g_{ik} \hat{D}^n \hat{D}_n - \hat{D}_i \hat{D}_k \right) \left[ Y_{(a)(b)} (\hat{D}_m \Phi^{(a)}) (\hat{D}^m \Phi^{(b)}) \right], \quad (45)$$

$$T_{ik}^{(V)} = Y_{(a)(b)} (\hat{D}_m \Phi^{(b)}) \left[ R_i^m (\hat{D}_k \Phi^{(a)}) + R_k^m (\hat{D}_i \Phi^{(a)}) \right] - \frac{1}{2} R_{ik} Y_{(a)(b)} (\hat{D}_m \Phi^{(a)}) (\hat{D}^m \Phi^{(b)}) - \\ - \frac{1}{2} \hat{D}^m \left\{ \hat{D}_i \left[ Y_{(a)(b)} (\hat{D}_m \Phi^{(a)}) (\hat{D}_k \Phi^{(b)}) \right] + \hat{D}_k \left[ Y_{(a)(b)} (\hat{D}_m \Phi^{(a)}) (\hat{D}_i \Phi^{(b)}) \right] - \right. \\ \left. - \hat{D}_m \left[ Y_{(a)(b)} (\hat{D}_i \Phi^{(a)}) (\hat{D}_k \Phi^{(b)}) \right] \right\} + \frac{1}{2} g_{ik} \hat{D}_m \hat{D}_n \left[ Y_{(a)(b)} \left( \hat{D}^m \Phi^{(a)} \right) \left( \hat{D}^n \Phi^{(b)} \right) \right]. \quad (46)$$

### III. EXACT SOLUTION OF THE WU-YANG TYPE OF THE NON-MINIMAL EYMH MODEL

#### A. Ansatz about the fields structure

Let the spacetime metric be of the spherically symmetric form

$$ds^2 = \sigma^2 N dt^2 - \frac{dr^2}{N} - r^2 (d\theta^2 + \sin^2 \theta d\varphi^2). \quad (47)$$

Here  $\sigma$  and  $N$  are functions depending on the radius  $r$  only and satisfying the asymptotic conditions

$$\sigma(\infty) = 1, \quad N(\infty) = 1. \quad (48)$$

We focus on the gauge field characterized by the special ansatz (see, [17]):

$$\mathbf{A}_0 = \mathbf{A}_r = 0,$$

$$\mathbf{A}_\theta = i\mathbf{t}_{(\varphi)}, \quad \mathbf{A}_\varphi = -i\nu \sin \theta \mathbf{t}_{(\theta)}. \quad (49)$$

The parameter  $\nu$  is a non-vanishing integer. The generators  $\mathbf{t}_{(r)}$ ,  $\mathbf{t}_{(\theta)}$  and  $\mathbf{t}_{(\varphi)}$  are the position-dependent ones and are connected with the standard generators of the SU(2) group as follows:

$$\begin{aligned} \mathbf{t}_{(r)} &= \cos \nu \varphi \sin \theta \mathbf{t}_{(1)} + \sin \nu \varphi \sin \theta \mathbf{t}_{(2)} + \cos \theta \mathbf{t}_{(3)}, \\ \mathbf{t}_{(\theta)} &= \partial_\theta \mathbf{t}_{(r)}, \quad \mathbf{t}_{(\varphi)} = \frac{1}{\nu \sin \theta} \partial_\varphi \mathbf{t}_{(r)}. \end{aligned} \quad (50)$$

The generators satisfy the relations

$$[\mathbf{t}_{(r)}, \mathbf{t}_{(\theta)}] = i \mathbf{t}_{(\varphi)}, \quad [\mathbf{t}_{(\theta)}, \mathbf{t}_{(\varphi)}] = i \mathbf{t}_{(r)}, \quad [\mathbf{t}_{(\varphi)}, \mathbf{t}_{(r)}] = i \mathbf{t}_{(\theta)}. \quad (51)$$

The field strength tensor

$$\mathbf{F}_{ik} = \partial_i \mathbf{A}_k - \partial_k \mathbf{A}_i + [\mathbf{A}_i, \mathbf{A}_k] \quad (52)$$

has only one non-vanishing component:

$$\mathbf{F}_{\theta\varphi} = i\nu \sin \theta \mathbf{t}_{(r)}, \quad (53)$$

which does not depend on the variable  $r$ . Our ansatz for the Higgs field is the following: we consider  $\Phi$  as a covariantly constant vector in the group space, which contains only one component, i.e.,

$$\Phi = \phi \mathbf{t}_{(r)}, \quad \hat{D}_k \Phi_{(a)} = \partial_k \Phi_{(a)} + \mathcal{G} \varepsilon_{(a)(b)(c)} A_k^{(b)} \Phi^{(c)} = 0. \quad (54)$$

This means that the Yang-Mills field strength tensor  $\mathbf{F}_{ik}$  and the Higgs field  $\Phi$  are parallel in the group space [18, 19] to the vector  $\mathbf{q} = \mathbf{t}_{(r)}$ . The Higgs equations are satisfied identically, when the field  $\phi$  is constant and coincides with  $v$ , the background value of the scalar field given by  $\Phi^{(a)} \Phi_{(a)} = v^2$ . Then the Yang-Mills equations (23) are satisfied identically also, and we deal with a new non-minimal Wu-Yang monopole solution, supplemented by the background covariantly constant Higgs field.

Finally, the equations for the gravity field are extremely simplified by the ansatz and its particular consequence  $X_{(a)(b)} F_{mn}^{(b)} = Q_1 F_{(a)mn}$ . In fact the equations for the gravity field for the model under discussion coincides with the ones, described in [6], if we replace the coupling parameters  $q_1, q_2, q_3$  by the  $Q_1 q_1, Q_1 q_2, Q_1 q_3$ , respectively. Thus, we can use the exact solutions of the Wu-Yang monopole type to the non-minimal Einstein-Yang-Mills model, as an exact solutions for the generalized non-minimal EYMH model with the covariantly constant Higgs field. The scalar field in this case does not modify the pressure and the energy of the system as whole, nevertheless, it play an important role, creating the privilege direction in the group space. In particular, the following solution for the non-minimal Wu-Yang monopole with regular center

$$\sigma(r) = 1, \quad N = 1 + \frac{r^2 (\kappa - 4Mr)}{2(r^4 + \kappa q)}, \quad (55)$$

is valid when  $Q_1 q_1 = -q, Q_1 q_2 = 4q, Q_1 q_3 = -6q$ , and  $q$  is positive.

#### IV. MULTI-METRIC REPRESENTATION OF THE MATERIAL TENSORS

By analogy with decomposition of material tensor in electrodynamics [20] the tensor  $C_{(a)(b)}^{ikmn}$  can be represented as

$$C_{(a)(b)}^{ikmn} = \frac{1}{2\hat{\mu}} \sum_{(\alpha)(\beta)(c)(d)} G_{(\alpha)(\beta)(a)(b)}^{(c)(d)} \left( g_{(c)}^{im(\alpha)} g_{(d)}^{kn(\beta)} - g_{(c)}^{in(\alpha)} g_{(d)}^{km(\beta)} \right), \quad (56)$$

with respect to color associated metrics  $g_{(c)}^{im(\alpha)}$ , where  $\hat{\mu}$  is some convenient factor [20]. Below we apply the elaborated formalism to the case, when the non-minimal Wu-Yang monopole accompanied by the constant scalar field forms the gravitational background for the moving particles, quasi-particles and waves.

### A. Associated metrics

In the spherically symmetric model under consideration the material tensor  $C_{(a)(b)}^{ikmn}$  reduces to

$$C_{(a)(b)}^{ikmn} = \delta_{(a)}^{(r)} \delta_{(b)}^{(r)} C_{(\text{long})}^{ikmn} + \left[ \delta_{(a)}^{(\theta)} \delta_{(b)}^{(\theta)} + \delta_{(a)}^{(\varphi)} \delta_{(b)}^{(\varphi)} \right] C_{(\text{trans})}^{ikmn}, \quad (57)$$

where  $C_{(\text{long})}^{ikmn}$  and  $C_{(\text{trans})}^{ikmn}$  are defined by (25) and (26), respectively. The reconstruction of the tensor  $C_{(\text{long})}^{ikmn}$  can be done using two longitudinal color metrics  $g_{(\text{long})}^{ik(A)}$  and  $g_{(\text{long})}^{ik(B)}$  as follows (see [20] for details)

$$C_{(\text{long})}^{ikmn} = \frac{1}{2\hat{\mu}_{(\text{long})}} \left\{ \left[ g_{(\text{long})}^{im(A)} g_{(\text{long})}^{kn(A)} - g_{(\text{long})}^{in(A)} g_{(\text{long})}^{km(A)} \right] - \right. \\ \left. - \gamma_{(\text{long})} \left[ \left( g_{(\text{long})}^{im(A)} - g_{(\text{long})}^{im(B)} \right) \left( g_{(\text{long})}^{kn(A)} - g_{(\text{long})}^{kn(B)} \right) - \left( g_{(\text{long})}^{in(A)} - g_{(\text{long})}^{in(B)} \right) \left( g_{(\text{long})}^{km(A)} - g_{(\text{long})}^{km(B)} \right) \right] \right\}, \quad (58)$$

where

$$g_{(\text{long})}^{im(A)} = U^i U^m + \frac{\varepsilon_{\perp}}{\varepsilon_{\parallel}} \Delta^{im} + \frac{(\varepsilon_{\perp} - \varepsilon_{\parallel})}{\varepsilon_{\parallel}} X_{\{r\}}^i X_{\{r\}}^m, \quad (59)$$

$$g_{(\text{long})}^{im(B)} = U^i U^m + \frac{\mu_{\perp}}{\mu_{\parallel}} \Delta^{im} + \frac{(\mu_{\perp} - \mu_{\parallel})}{\mu_{\parallel}} X_{\{r\}}^i X_{\{r\}}^m. \quad (60)$$

Mention that  $g_{(\text{long})}^{ik(B)}$  can be obtained from  $g_{(\text{long})}^{ik(A)}$  by a formal replacement of the symbol  $\varepsilon$  by the symbol  $\mu$ . We used the following notations

$$\varepsilon_{\perp} \equiv \varepsilon_{\theta}^{\theta} = \varepsilon_{\varphi}^{\varphi} = 1 + 2Q_1 \mathcal{R}_{\theta t}^{\theta t} = 1 + 2Q_1 \mathcal{R}_{\theta r}^{\theta r} = (\mu^{-1})_{\theta}^{\theta} = (\mu^{-1})_{\varphi}^{\varphi} \equiv \frac{1}{\mu_{\perp}}, \quad (61)$$

$$\varepsilon_r^r \equiv \varepsilon_{\parallel} = 1 + 2Q_1 \mathcal{R}_{rt}^{rt}, \quad (\mu^{-1})_r^r \equiv \frac{1}{\mu_{\parallel}} = 1 + 2Q_1 \mathcal{R}_{\theta\varphi}^{\theta\varphi}, \quad (62)$$

$$\nu_m^k = 0, \quad \frac{1}{\hat{\mu}_{(\text{long})}} = \varepsilon_{\parallel}, \quad \frac{1}{\gamma_{(\text{long})}} = 1 - \frac{\varepsilon_{\parallel} \mu_{\perp}}{\varepsilon_{\perp} \mu_{\parallel}}, \quad (63)$$

$$U^k = \delta_t^k \frac{1}{\sqrt{N(r)}}, \quad X_{\{r\}}^k = \delta_r^k \sqrt{N(r)}, \quad (64)$$

$$\mathcal{R}_{r\theta}^{r\theta} = \mathcal{R}_{r\varphi}^{r\varphi} = \mathcal{R}_{\theta t}^{\theta t} = \mathcal{R}_{\varphi t}^{\varphi t} = -\frac{r}{(r^4 + a^4)^3} [6qMr^8 - 3a^4 r^7 - 24qMa^4 r^4 + 5a^8 r^3 + 2qMa^8], \quad (65)$$

$$\mathcal{R}_{rt}^{rt} = \frac{r}{(r^4 + a^4)^3} [12qMr^8 - 7a^4 r^7 - 76qMa^4 r^4 + 17a^8 r^3 + 8qMa^8], \quad (66)$$

$$\mathcal{R}_{\theta\varphi}^{\theta\varphi} = \frac{r^4}{(r^4 + a^4)^3} [12qMr^5 - 5a^4 r^4 - 20qMa^4 r + 3a^8]. \quad (67)$$

A new parameter  $a$  with dimensionality of length is defined as follows:  $a^4 = \kappa q$ , where  $\kappa = 8\pi\nu^2/\mathcal{G}^2$ . To obtain the decomposition for  $C_{(\text{trans})}^{ikmn}$  one can simply replace the mark (long) by (trans) and then replace  $Q_1$  by one. Clearly, this simple model involves the following non-trivial components of the tensor  $G_{(\alpha)(\beta)(a)(b)}^{(c)(d)}$ , introduced by (56):

$$G_{(\alpha)(\beta)(\theta)(\theta)}^{(\theta)(\theta)} = G_{(\alpha)(\beta)(\varphi)(\varphi)}^{(\varphi)(\varphi)} \neq G_{(\alpha)(\beta)(r)(r)}^{(r)(r)}, \quad (68)$$



with  $(\alpha), (\beta) = (A), (B)$ , which satisfy the linear relations

$$G_{(A)(A)(a)(b)}^{(c)(d)} + G_{(B)(A)(a)(b)}^{(c)(d)} = 1, \quad G_{(A)(B)(a)(b)}^{(c)(d)} + G_{(B)(B)(a)(b)}^{(c)(d)} = 0,$$

$$G_{(A)(B)(r)(r)}^{(r)(r)} = G_{(B)(A)(r)(r)}^{(r)(r)} = \gamma_{(\text{long})},$$

$$G_{(A)(B)(\theta)(\theta)}^{(\theta)(\theta)} = G_{(B)(A)(\theta)(\theta)}^{(\theta)(\theta)} = G_{(A)(B)(\varphi)(\varphi)}^{(\varphi)(\varphi)} = G_{(B)(A)(\varphi)(\varphi)}^{(\varphi)(\varphi)} = \gamma_{(\text{trans})}. \quad (69)$$

Thus, in order to reconstruct the linear response tensor  $C_{(a)(b)}^{ikmn}$  for the model under discussion one needs four associated metrics  $g_{(\text{long})}^{im(A)}, g_{(\text{long})}^{im(B)}, g_{(\text{trans})}^{im(A)}$  and  $g_{(\text{trans})}^{im(B)}$ . When  $Q_1 = 1$ , i.e., there is no privilege direction in the group space, the corresponding longitudinal and transversal  $A$  and  $B$  metrics coincide.

### B. Color metrics

The associated metrics  $g_{(\text{long})}^{im(A)}, g_{(\text{long})}^{im(B)}, g_{(\text{trans})}^{im(A)}$  and  $g_{(\text{trans})}^{im(B)}$  are the color ones. To prove this statement we have to check two facts. First, the equalities

$$g_{(\text{long})}^{km(A)} p_k p_m = 0, \quad \text{or} \quad g_{(\text{long})}^{km(B)} p_k p_m = 0, \quad (70)$$

are satisfied in the WKB-approximation for  $A$  and  $B$ -waves, respectively, when the Yang-Mills potential has a form  $A_k^{(r)} \mathbf{t}_{(r)}$ . Second, the equalities

$$g_{(\text{trans})}^{km(A)} p_k p_m = 0, \quad \text{or} \quad g_{(\text{trans})}^{km(B)} p_k p_m = 0, \quad (71)$$

hold, when the Yang-Mills potential is of the form  $A_k^{(\theta)} \mathbf{t}_{(\theta)} + A_k^{(\varphi)} \mathbf{t}_{(\varphi)}$ . Indeed, let us consider the Yang-Mills equations in the leading order WKB-approximation:

$$C_{(a)(b)}^{ikmn} p_k p_m A_n^{(b)} = 0, \quad (72)$$

with the tensor  $C_{(a)(b)}^{ikmn}$  given by (57), (25), (26). When the group index  $(a)$  coincides with  $(r)$ , one obtains

$$C_{(r)(r)}^{ikmn} p_k p_m A_n^{(r)} \equiv C_{(\text{long})}^{ikmn} p_k p_m A_n^{(r)} = 0. \quad (73)$$

When  $(a) = (\theta)$  or  $(a) = (\varphi)$ , the system (72) gives two equations for the transversal components of the Yang-Mills potential  $A_n^{(\theta)}$  and  $A_n^{(\varphi)}$

$$C_{(\text{trans})}^{ikmn} p_k p_m A_n^{(a)} = 0. \quad (74)$$

In the WKB approximation the gauge potentials  $A_k^{(a)}$  and the field strength  $F_{kl}^{(a)}$  can be extrapolated as follows

$$A_k^{(a)} \rightarrow \mathcal{A}_k^{(a)} e^{i\Psi}, \quad F_{kl}^{(a)} \rightarrow i \left[ p_k \mathcal{A}_l^{(a)} - p_l \mathcal{A}_k^{(a)} \right] e^{i\Psi}, \quad p_k = \nabla_k \Psi. \quad (75)$$

Mention that the nonlinear terms in (6) give the values of the next order in WKB approximation, thus, such a model of gauge field is effectively Abelian. In the leading order approximation the Yang-Mills equations reduce to

$$C_{(a)(b)}^{ikmn} p_k p_m \mathcal{A}_n^{(b)} = 0. \quad (76)$$

For the sake of simplicity and in this subsection only we omit the marks (long) and (trans) in the associated metrics and in the parameters  $\mu$  and  $\gamma$ , since the results are similar for the longitudinal and transversal cases. Substitution  $C_{(a)(b)}^{ikmn}$  from (58) with  $g^{ik(A)}$  from (59) and  $g^{ik(B)}$  from (60) with  $\hat{\mu}$  and  $\gamma$  given by (63) yields

$$g^{im(A)} p_m \left[ g^{kn(A)} p_k \mathcal{A}_n^{(a)} \right] - g^{in(A)} \mathcal{A}_n^{(a)} \left[ g^{km(A)} p_k p_m \right] =$$

$$= \frac{1}{\varepsilon_{||}^2 \mu_{\perp}^2} \left( 1 - \frac{\varepsilon_{||} \mu_{\perp}}{\varepsilon_{\perp} \mu_{||}} \right) p_k p_m \mathcal{A}_n^{(a)} \left[ \left( X_{(1)}^i X_{(2)}^k - X_{(2)}^i X_{(1)}^k \right) \left( X_{(1)}^m X_{(2)}^n - X_{(2)}^m X_{(1)}^n \right) \right], \quad (77)$$

where  $X_{\{1\}}^i$ ,  $X_{\{2\}}^i$  are unit spacelike vectors, orthogonal to  $U^i$ ,  $X_{\{r\}}^i$ , and each other. Projection of this equations onto the velocity four-vector  $U_i$  gives the scalar ratio

$$(U^m p_m) \left[ g^{kn(A)} p_k \mathcal{A}_n^{(a)} \right] = \left( U^n \mathcal{A}_n^{(a)} \right) \left[ g^{km(A)} p_k p_m \right], \quad (78)$$

which is satisfied if, for instance, we use the Landau gauge  $U^n \mathcal{A}_n^{(a)} = 0$  and the condition of orthogonality of the wave four-vector and amplitude four-vector in the first associated metric, i.e.,  $g^{kn} p_k \mathcal{A}_n^{(a)} = 0$ . Projections onto the axes, given by  $X_{\{1\}}^i$ ,  $X_{\{2\}}^i$  and  $X_{\{r\}}^i$  yield, respectively,

$$\begin{aligned} \mathcal{A}_{\{1\}}^{(a)} \left[ g^{km(A)} p_k p_m + p_{\{2\}}^2 \left( \frac{1}{\varepsilon_{||} \mu_{\perp}} - \frac{1}{\varepsilon_{\perp} \mu_{||}} \right) \right] - \mathcal{A}_{\{2\}}^{(a)} \left[ p_{\{1\}} p_{\{2\}} \left( \frac{1}{\varepsilon_{||} \mu_{\perp}} - \frac{1}{\varepsilon_{\perp} \mu_{||}} \right) \right] &= 0, \\ \mathcal{A}_{\{1\}}^{(a)} \left[ p_{\{1\}} p_{\{2\}} \left( \frac{1}{\varepsilon_{||} \mu_{\perp}} - \frac{1}{\varepsilon_{\perp} \mu_{||}} \right) \right] - \mathcal{A}_{\{2\}}^{(a)} \left[ g^{km(A)} p_k p_m + p_{\{1\}}^2 \left( \frac{1}{\varepsilon_{||} \mu_{\perp}} - \frac{1}{\varepsilon_{\perp} \mu_{||}} \right) \right] &= 0, \\ \mathcal{A}_{\{r\}}^{(a)} \left[ g^{km(A)} p_k p_m \right] &= 0, \end{aligned} \quad (79)$$

where  $\mathcal{A}_{\{1\}}^{(a)} \equiv X_{\{1\}}^k \mathcal{A}_k^{(a)}$ ,  $p_{\{1\}} \equiv X_{\{1\}}^k p_k$ , etc. Taking into account the relation

$$\left[ g^{km(B)} p_k p_m \right] = \left[ g^{km(A)} p_k p_m \right] + \left( p_{\{1\}}^2 + p_{\{2\}}^2 \right) \left( \frac{1}{\varepsilon_{||} \mu_{\perp}} - \frac{1}{\varepsilon_{\perp} \mu_{||}} \right), \quad (80)$$

one can conclude that nontrivial solution of (79) exists, when

$$\left[ g^{km(A)} p_k p_m \right] \left[ g^{lj(B)} p_l p_j \right] = 0. \quad (81)$$

Thus, the associated metrics  $g^{km(A)}$  and  $g^{km(B)}$  are the color metrics, which indicate both longitudinal and transversal ones.

### C. Color-acoustic metrics

Tensor  $\mathcal{C}_{(a)(b)}^{ik}$  (32) can be associated with two metrics  $\tilde{g}_{(\text{long})}^{ik}$  and  $\tilde{g}_{(\text{trans})}^{ik}$ :

$$\mathcal{C}_{(a)(b)}^{ik} = \left[ \delta_{(a)}^{(\theta)} \delta_{(b)}^{(\theta)} + \delta_{(a)}^{(\varphi)} \delta_{(b)}^{(\varphi)} \right] \tilde{g}_{(\text{trans})}^{ik} + \tilde{g}_{(\text{long})}^{ik} \delta_{(a)}^{(r)} \delta_{(b)}^{(r)}. \quad (82)$$

In the leading order of the WKB approximation, the propagation of the scalar particle with the color index  $(a) = (r)$ , described by the potential  $\Phi^{(r)} \mathbf{t}_{(r)}$ , is equivalent to the motion in the effective spacetime with the metric  $\tilde{g}_{(\text{long})}^{ik}$ . As for the particle, described the potential  $\Phi^{(\theta)} \mathbf{t}_{(\theta)}$  or  $\Phi^{(\varphi)} \mathbf{t}_{(\varphi)}$ , the corresponding effective spacetime has a metric  $\tilde{g}_{(\text{trans})}^{ik}$ . We obtain *color* birefringence for scalar waves (quasi-particles), since the velocity of the wave depends on its color. Taking into account the formulas

$$\Re_t^t = \Re_r^r = \frac{1}{2}(2q_4 + q_5)N'' + \frac{1}{r}(4q_4 + q_5)N' + \frac{2}{r^2}q_4(N - 1), \quad (83)$$

$$\Re_{\theta}^{\theta} = \Re_{\varphi}^{\varphi} = q_4 N'' + \frac{1}{r}(4q_4 + q_5)N' + \frac{1}{r^2}(2q_4 + q_5)(N - 1), \quad (84)$$

we can reconstruct the color-acoustic metric for arbitrary parameters  $Q_2$ ,  $M$ ,  $q_4$  and  $q_5$ . Nevertheless, to complete the illustration of the possibilities of the effective metric approach we restrict ourselves by the model with  $M = 0$ . Then the longitudinal color-acoustic metric is

$$\tilde{g}_{(\text{long})}^{ik} = \mathcal{A}^2 \left\{ \left[ \frac{1}{N} \delta_t^i \delta_t^k - N \delta_r^i \delta_r^k \right] - \frac{1}{r^2 \mathcal{B}^2} \left[ \delta_{\theta}^i \delta_{\theta}^k + \frac{1}{\sin^2 \theta} \delta_{\varphi}^i \delta_{\varphi}^k \right] \right\}, \quad (85)$$

where the conformal factor  $\mathcal{A}^2$  is

$$\mathcal{A}^2 = 1 + \frac{Q_2}{2q(\xi+1)^3} [\xi^2 q_5 - 4\xi(5q_4 + 3q_5) + 3(4q_4 + q_5)] , \quad (86)$$

and the angular factor  $\mathcal{B}^2$  is given by

$$\mathcal{B}^2 = \frac{2q(\xi+1)^3 + Q_2 [\xi^2 q_5 - 4\xi(5q_4 + 3q_5) + 3(4q_4 + q_5)]}{2q(\xi+1)^3 + Q_2 [-\xi^2 q_5 - 2\xi(10q_4 - q_5) + 3(4q_4 + q_5)]} . \quad (87)$$

Here we introduced a new convenient dimensionless variable  $\xi = r^4/\kappa q$ , which clearly takes non-negative values only. Again, the transversal metric can be obtained from this formulas after the formal replacement  $Q_2 \rightarrow 1$ .

## V. DISCUSSION

The formulas (59) and (60) present the associated metrics for the non-minimal EYM model under discussion. As well, in the WKB approximation the Yang-Mills equations are satisfied when (70) or (71) are valid. Thus, these associated metrics can be interpreted as color ones. This means, in particular, that the propagation of a test Yang-Mills wave in the non-minimally active spacetime can be described by two dispersion relations

$$\omega_{(A)}^2 = \frac{p_\perp^2}{\varepsilon_{||}\mu_\perp} + p_{||}^2, \quad \omega_{(B)}^2 = \frac{p_\perp^2}{\varepsilon_\perp\mu_{||}} + p_{||}^2, \quad (88)$$

where

$$\omega = p_t U^t, \quad p_\perp^2 = -(p_\theta p^\theta + p_\varphi p^\varphi), \quad p_{||}^2 = -p_r p^r. \quad (89)$$

When the wave propagates in the radial direction, i.e.,  $p_\perp = 0$ , or in the transverse one, i.e.,  $p_{||} = 0$ , the corresponding phase velocities can be easily found as  $\mathcal{V}_{||} = \omega/p_{||}$ ,  $\mathcal{V}_\perp = \omega/p_\perp$ . For A-wave and B-wave they are, respectively,

$$\mathcal{V}_{||}^{(A)} = \mathcal{V}_{||}^{(B)} = 1, \quad \mathcal{V}_\perp^{(A)} = \frac{1}{\sqrt{\varepsilon_{||}\mu_\perp}}, \quad \mathcal{V}_\perp^{(B)} = \frac{1}{\sqrt{\varepsilon_\perp\mu_{||}}}, \quad (90)$$

i.e., both the longitudinal ( $Q_1 \neq 1$ ) and transversal ( $Q_1 = 1$ ) color waves, propagating radially, have the phase velocity equal to the speed of light in the minimal vacuum.

As an illustration, consider now a specific particular case of the exact solution (55) characterized by  $M = 0$ . This solution does not admit singularities of the horizon type, since  $N > 1$  everywhere. It is convenient to present the basic formulas for the velocities of the waves propagating in the direction, orthogonal to the radius, as follows

$$\left(\mathcal{V}_\perp^{(A)}(\xi)\right)^2 = \frac{1}{\varepsilon_{||}\mu_\perp} = \frac{\xi^3 + (3 + 6Q_1)\xi^2 + (3 - 10Q_1)\xi + 1}{\xi^3 + (3 - 14Q_1)\xi^2 + (3 + 34Q_1)\xi + 1}, \quad (91)$$

$$\left(\mathcal{V}_\perp^{(B)}(\xi)\right)^2 = \frac{1}{\varepsilon_\perp\mu_{||}} = \frac{\xi^3 + (3 - 10Q_1)\xi^2 + (3 + 6Q_1)\xi + 1}{\xi^3 + (3 + 6Q_1)\xi^2 + (3 - 10Q_1)\xi + 1}. \quad (92)$$

For these formulas one can obtain the transversal color wave velocity when  $Q_1 = 1$ , as well as, for longitudinal one, say, when  $Q_1 = 2$ . When do the phase velocities coincide with the speed of light in minimal vacuum, i.e., when do they equal to one? The following facts clarify this question.

- (i)  $\mathcal{V}_\perp^{(A)}(\infty) = \mathcal{V}_\perp^{(B)}(\infty) = 1$  for arbitrary  $Q_1$ .
- (ii)  $\mathcal{V}_\perp^{(A)}(0) = \mathcal{V}_\perp^{(B)}(0) = 1$  for arbitrary  $Q_1$ .
- (iii)  $\mathcal{V}_\perp^{(A)}(\frac{11}{9}) = 1$  for arbitrary  $Q_1$ .
- (iv)  $\mathcal{V}_\perp^{(B)}(1) = 1$  for arbitrary  $Q_1 \neq 2$ .
- (v)  $\left(\mathcal{V}_\perp^{(A)}(\xi)\right)^2 > 0$  for any  $\xi$ , when  $-0.241 < Q_1 < 8/9$ .
- (vi)  $\left(\mathcal{V}_\perp^{(B)}(\xi)\right)^2 > 0$  for any  $\xi$ , when  $-54/25 < Q_1 < 8/9$ .

The behavior of the functions  $\left(\mathcal{V}_{\perp}^{(A)}(\xi)\right)^2$  and  $\left(\mathcal{V}_{\perp}^{(B)}(\xi)\right)^2$  depend on the value of the parameter  $Q_1$ . Remember, that the value  $Q_1 = 1$  relates to the transversal color waves, and  $Q_1 \neq 1$  relates to the longitudinal one. In order to clarify the principal moments of analysis we put further  $Q_1 = 2$  for the longitudinal case. Let us attract the attention to two details. The first one is connected with the nulls of the functions under consideration, the second relates to the infinite value. Both points relate to the singularities of the color metrics. When the functions (91) and (92) become negative, we deal with *inaccessibility zones*.

#### Longitudinal A-wave

The function  $\left(\mathcal{V}_{\perp}^{(A)}(\xi)\right)^2$  for  $Q_1 = 2$  is negative for the ranges  $\sqrt{65} - 8 < \xi < 1$  and  $3.283 < \xi < 21.731$ . In this ranges of radial variable  $\xi = r^4/\kappa q$  the propagation of the color waves of this type is impossible. At  $r = r_1 \equiv (\kappa q(\sqrt{65} - 8))^{1/4}$  and  $r = r_2 \equiv (\kappa q)^{1/4}$  the phase velocity vanishes and the wave stops. At  $r = 3.283$  and  $r = 21.731$  the phase velocity is infinite. Admissible ranges of  $\xi$  for the wave propagation are, therefore,  $0 < \xi < \sqrt{65} - 8$ ,  $1 < \xi < 3.283$  and  $21.731 < \xi < \infty$ . The first admissible range is of finite length, the corresponding part of the space can be indicated as *first inner admissible region* or *trapped region*. This trapped region is arranged between two spheres of the zeroth radius and of the radius  $r_1$ . On the left edge of this region ( $r = 0$ ) the phase velocity is equal to one relating to the absence of classical gravity force ( $N = 1, N' = 0$ ). With increasing of radius the phase velocity decreases monotonically and vanishes at the right edge of the trapped region, thus, the transversally propagating wave stops at this surface. The second (*inner*) admissible region can also be indicated as trapped one, nevertheless, the phase velocity at the left edge,  $\xi = 1$ , is vanishing, whereas the right edge is characterized by the infinite phase velocity, passing the value equal to one at  $\xi = 11/5$ . The third admissible range is infinite. The phase velocity is infinite at the left edge of this region and tends monotonically to one at infinity. The non-minimal radius  $r = a$  indicates the boundary between the *first inaccessibility zone* and second admissible region. The *second inaccessibility zone*  $3.283 < \xi < 21.731$  is characterized by the infinite phase velocity barriers at the left and right edges.

#### Longitudinal B-wave

When  $Q_1 = 2$  the formula (92) reduces to

$$\left(\mathcal{V}_{\perp}^{(B)}(\xi)\right)^2 = \frac{\xi^2 - 16\xi - 1}{\xi^2 + 16\xi - 1}, \quad (93)$$

i.e., the denominator and numerator are presented by the polynomials quadratic in  $\xi$ . Now this function is discontinuous at  $\xi = \sqrt{65} - 8$ , this point being the right edge of the first admissible region,  $0 < \xi < \sqrt{65} - 8$ , which is the trapped region of the second type. When the radius grows the phase velocity increases from one to infinity, and a wave does not stop at the boundary, but is, on the contrary, infinite. The range  $\sqrt{65} - 8 < \xi < \sqrt{65} + 8$  relates to the inaccessible zone, since  $\left(\mathcal{V}_{\perp}^{(B)}(\xi)\right)^2$  is negative. The second admissible region is  $\sqrt{65} + 8 < \xi < \infty$ , the phase velocity is equal to zero at  $\xi = \sqrt{65} + 8$  and tends to one asymptotically at  $\xi \rightarrow \infty$ .

#### Transversal A-wave

When  $Q_1 = 1$ , the curve  $\left(\mathcal{V}_{\perp}^{(A)}(\xi)\right)^2$  has no discontinuity. The trapped region of the first type is situated at  $0 < \xi < 0.191$ ; the second admissible region is arranged at  $0.539 < \xi < \infty$ ; curve passes one at  $\xi = 11/5$ , then reaches the maximal value and tends to one at  $\xi \rightarrow \infty$ . The inaccessibility zone is  $0.191 < \xi < 0.539$ .

#### Transversal B-wave

For this wave one has two trapped regions, two inaccessibility zones and one infinite admissible region. The first trapped region is characterized by the inequality  $0 < \xi < 0.191$ , the phase velocity starts from one at the left edge and reaches infinity at the right edge. The second one is at  $0.539 < \xi < 1.853$ , the phase velocity starts from infinity at the left edge and vanished at the right one. The first inaccessibility zone is arranged at  $0.191 < \xi < 0.539$ , the phase velocity is infinite at both edges. The second one is characterized by  $1.853 < \xi < 5.249$ , the phase velocity being vanishing at both edges. The infinite admissible region is  $5.249 < \xi < \infty$ . The phase velocity starts from zero at the left edge and tends monotonically to one at  $\xi \rightarrow \infty$ .

#### Color-acoustic waves

In order to complete our illustration let us consider a particular model with the following parameters:

$$q_5 = -\frac{q}{9}, \quad q_4 = \frac{q}{36}. \quad (94)$$

Our choice relates to the idea that non-minimal radius  $a = (\kappa q)^{1/4}$ , appeared in the equations for color gauge waves, is the same one for the color-acoustic waves, i.e., there is no additional non-minimal Constant of Nature (see, e.g.,

[21]). With (94) the functions  $\mathcal{A}^2(\xi)$  and  $\mathcal{B}^2(\xi)$  are, respectively,

$$\mathcal{A}^2 = \frac{18(\xi + 1)^3 + Q_2\xi(7 - \xi)}{18(\xi + 1)^3}, \quad (95)$$

$$\mathcal{B}^2 = \frac{18(\xi + 1)^3 + Q_2\xi(7 - \xi)}{18(\xi + 1)^3 - Q_2\xi(7 - \xi)}, \quad (96)$$

(see formulas (86) and (87)). The toy models with  $Q_2 = 2$  (longitudinal case) and  $Q_2 = 1$  (transversal case) are non-singular. Indeed, in both cases the cubic polynomials in the numerators and denominators of (95) and (96) have no real positive roots. This means that both functions,  $\mathcal{A}^2(\xi)$  and  $\mathcal{B}^2(\xi)$  are continuous and positive for  $\xi > 0$ . In addition, both functions, start from one at  $\xi = 0$ , pass through one at  $\xi = 7$  and tend to one at  $\xi \rightarrow \infty$ . As well, the function  $\mathcal{B}^2(\xi)$  reaches the maximum at  $\xi = 8 - \sqrt{57}$  and minimum at  $\xi = 8 + \sqrt{57}$ . Thus, in contrast to the case of color waves the color-acoustic waves in this toy model have neither trapped regions, nor inaccessible zones. The phase velocities of the corresponding color-acoustic waves are non-monotonic functions of the radius, nevertheless, they can reach neither zero value, nor infinity.

#### Remark

In general case the propagation direction of the color or color-acoustic wave is not pure longitudinal or pure transversal with respect to the radial direction. In general case the trajectory of the corresponding quasi-particle becomes very sophisticated depending on the value of the impact parameter. This requires a detailed numerical analysis, and we hope to discuss that results in a separate paper.

#### Alternative description

Particle dynamics can be alternatively described using the equation of motion with effective force. Instead of equation of null geodesics in the effective spacetime with metric  $g_{(h)}^{ik(\alpha)}$ , where the upper index ( $\alpha$ ) indicates the  $A$  and  $B$  cases in the classification of effective metrics, the lower index ( $h$ ) is treated as (long) and (trans), one can write the equation

$$\frac{d^2 x^i}{d\tau^2} + \Gamma_{kl}^i \frac{dx^k}{d\tau} \frac{dx^l}{d\tau} = F_{(h)}^{i(\alpha)}, \quad (97)$$

where

$$F_{(h)}^{i(\alpha)} \equiv \Pi_{kl(h)}^{i(\alpha)} \frac{dx^k}{d\tau} \frac{dx^l}{d\tau}, \quad \Pi_{kl(h)}^{i(\alpha)} \equiv \Gamma_{kl}^i - \Gamma_{kl(h)}^{i(\alpha)}. \quad (98)$$

$\Gamma_{kl}^i$  and  $\Gamma_{kl(h)}^{i(\alpha)}$  are the Christoffel symbols for the real and effective spacetimes, respectively. The quantity  $\Pi_{kl(h)}^{i(\alpha)}$ , the difference of the Christoffel symbols, symmetric with respect to indices  $k$  and  $l$ , is known to be a tensor, thus, the quantity  $F_{(h)}^{i(\alpha)}$  is a vector. Since we consider the interaction of Yang-Mills and Higgs field with spacetime curvature, we can indicate this force as a tidal one. The tidal force  $F_{(h)}^{i(\alpha)}$  is quadratic in the particle four-velocity and is predetermined by the structure of the tensor  $\Pi_{kl(h)}^{i(\alpha)}$ .

For the Wu-Yang model the tensor  $\Pi_{kl(\text{long})}^{i(A)}$  has the following non-vanishing components:

$$\begin{aligned} \Pi_{\theta\theta(\text{long})}^{r(A)} &= \Pi_{\varphi\varphi(\text{long})}^{r(A)} / \sin^2 \theta = \frac{1}{2} N(r) \frac{d}{dr} \left[ r^2 \left( \frac{\varepsilon_{||}}{\varepsilon_{\perp}} - 1 \right) \right], \\ \Pi_{\theta r(\text{long})}^{\theta(A)} &= \Pi_{\varphi r(\text{long})}^{\varphi(A)} = \frac{1}{2} \frac{d}{dr} \left[ \ln \left( \frac{\varepsilon_{||}}{\varepsilon_{\perp}} \right) \right]. \end{aligned} \quad (99)$$

The non-vanishing components of the tensor  $\Pi_{kl(\text{long})}^{i(B)}$  can be obtained from (99) by the formal replacement of the symbol  $\varepsilon$  by the symbol  $\mu$ . As for  $\Pi_{kl(\text{trans})}^{i(A)}$  and  $\Pi_{kl(\text{trans})}^{i(B)}$ , they can be obtained from the corresponding longitudinal components if we put  $Q_1 \rightarrow 1$ . When the particle moves radially, i.e.,  $\frac{d\theta}{d\tau} = \frac{d\varphi}{d\tau} = 0$ , the tidal force vanishes. This agrees with the result obtained above that the radially propagating color waves have the phase velocity equal to the speed of light in the standard (minimal) vacuum.

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